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ROTOR BLADE VIBRATION MODES AND FREQUENCIES

SUMMARY

The primary object of the research has been the numerical evaluation of certain exact solutions of the differential equation of motion of a flexible rotor blade under the action of centrifugal force and inertia, giving the natural frequencies of free vibration and the deflection curves or modes of the different possible types of vibration. These numerical results apply to various degrees and kinds of blade taper, and they have been tabulated and plotted. The taper range is

- (B_0) Uniform mass distribution along the blade radius
- (D_0) Linear (triangular) taper
- (A) Elliptical taper
- (B) Generalized elliptical taper
- (C) Parabolic taper
- (D) Generalized parabolic taper
- (E) Blade extension, parabolic type taper
- (E') Hyperbolic taper

(B_0) can be regarded as a special case of (B) or (D) and (D_0) of (D), while (E) is simply (D) extended across the axis of rotation. (E') is the limiting case of (E).

From the results for the mathematical taper forms listed, the frequencies of the various modes for non-mathematical taper forms, as used in most actual blades, can be interpolated fairly accurately according to the degree of taper, specified by the distance of the blade centroid from the axis, in terms of the tip radius. In interpolating modes (deflections) some regard should also be paid to the type of taper, e.g. whether concave or convex.

Blade root conditions influence the modes and frequencies. In general these can be covered by the location of the flapping hinge, or the equivalent hinge position for built-in blade roots. Hinge location is one of the variables included in the calculations, so that its effect, which is not very sensitive to taper, can easily be found.

For actual blades possessing flexural rigidity a well-known formula for vibration frequency is $\Omega^2 = (k^2w^2 + k_B^2)$ where Ω is the actual resultant frequency, (kw) the flexible-blade frequency considered in this report, and k_B the frequency of the non-rotating blade vibrating as a beam. The present work shows that the formula is exact for a certain "standard" distribution of bend rigidity (E_I) along the blade radius, the bending mode then being identical with the corresponding mode for a flexible rotating blade having the same mass taper. Even when not exact the formula represents a useful analysis of the frequency showing the results of a change in r.p.m. or flexural rigidity, and checking more detailed calculations.

Arising from the present work, although not part of the original programme and therefore not dealt with in any detail in the report:

(i) The response of a rotating blade to periodic, sudden or, as a special case, static loading can be calculated exactly for any of the tapered blades having the "standard" stiffness distribution. This provides a method of stressing the blade.

(ii) Alternatively, independent exact solutions are available for the static loading case, cognate with those for the vibrating blade.

(iii) Applicable not only to the taper forms specified above, but also to the wider taper range obtained by truncating these forms, complete and exact solutions are available for vibrating beams having a wide range of mass and stiffness taper.

(iv) Shear deflection effects can readily be included in the calculations either for the non-rotating beams or the rotating blades.

(v) Truncated uniform or tapered blades carrying tip masses are covered by a fairly simple extension of the solution for the intact blade.

(vi) The differential equation for a rotor blade, whether statically loaded or vibrating is mathematically identical with that for a so-called box wing consisting of two spars and a torsion box. Solutions and results for such wings can therefore be deduced from those given here.

LIST OF SYMBOLS

x	Blade station (fraction of tip radius, R)
y	Blade deflection
w	Angular velocity of rotor
p	Blade frequency
k	Blade frequency ratio p/w
m, n	indices
g	Acceleration due to gravity
P	Centrifugal tension in blade
a	Coefficient
EI	Blade bend rigidity
GA	Blade shear rigidity

Reference: Aircraft Engineering, November 1958

SUMMARY OF FREQUENCIES

Values of $k^2 = p^2/w^2$

Case	Mass Distribution	Centroid	Mode 0	1s	1a	2s	2a	3s	3a
B_0	Uniform	0·5	1	3	6	10	15	21	28
B	$(1 - x^2)^{\frac{1}{4}}$	0·456	1	2·8	5·4	8·8	13	18	23·8
E	$(1 - x)^{-\frac{1}{2}}$	0·455	1		5 38		13·3		
A	$(1 - x^2)^{\frac{1}{2}}$	0·424	1	2·67	5	8	11·67	16	21
E	$(1 - x)^{-\frac{3}{4}}$	0·404	1		5·05		12·2		
C	$(1 - x)^{\frac{1}{2}}$	0·40	1		4·75		11·07		
B	$(1 - x^2)$	0·375	1	2·5	4·5	7	10	13·5	17·5
D_0	$(1 - x)$	0·333	1		4·2		9·2		15·9
E'	x^{-1}	0	1		4		9		16

y as a function of x and k for uniform blade

ANTI-SYMMETRICAL MODES

FREQ k^2 / STA $1-x$	0th mode			1th symm. node			1th mode			2nd mode				3rd mode		
	1·1	1·3	1·5	3	4	5	6	7	8	15	16	18	20	28	33	36
1	-0.064	-0.176	-0.268	-0.500	-0.389	-0.199	0	0.174	0.298	0	-0.082	-0.240	-0.308	0	0.234	0.273
0.95	-0.019	-0.125		-0.496			-0.080	0.100	0.244	0.099	0.007					
0.9	0.038	-0.073	-0.168	-0.485	-0.449	-0.315	-0.149	0.016	0.164	0.179	0.102	-0.062	-0.200	-0.194	0.052	0.140
0.85	0.089	-0.019			-0.463		-0.215	-0.054	0.078	0.254	0.188	0.032				
0.8	0.141	0.035	-0.059	-0.440	-0.472	-0.403	-0.280	-0.130	0.018	0.307	0.259	0.126	-0.014	-0.295	-0.168	-0.042
0.7	0.246	0.146	0.054	-0.365	-0.456	-0.450	-0.377	-0.279	-0.157	0.342	0.335	0.275	0.175	-0.268	-0.284	-0.239
0.6	0.351	0.260	0.176	-0.260	-0.398	-0.450	-0.440	-0.389	-0.307	0.270	0.307	0.335	0.315	-0.023	-0.207	-0.268
0.5	0.458	0.377	0.301	-0.125	-0.295	-0.393	-0.438	-0.434	-0.409	0.089	0.151	0.257	0.304	0.212	0.038	-0.074
0.4	0.565	0.497	0.432	0.040	-0.142	-0.273	-0.363	-0.412	-0.431	-0.152	-0.091	0.026	0.100	0.344	0.279	0.212
0.3	0.673	0.619	0.568	0.235	0.061	-0.081	-0.193	-0.278	-0.342	-0.365	-0.332	-0.254	-0.169	0.165	0.271	0.307
0.2	0.781	0.744	0.708	0.460	0.317	0.191	0.080	-0.017	-0.100	-0.400	-0.410	-0.417	-0.404	-0.230	-0.099	-0.017
0.15			0.779				0.261			-0.292	-0.319	-0.367	-0.398			
0.1	0.890	0.871	0.852	0.715	0.630	0.549	0.472	0.400	0.332	-0.006	-0.082	-0.153	-0.214	-0.368	-0.404	-0.410
0.05			0.925	0.854	0.808	0.763	0.719		0.634	0.373	0.339	0.281	0.217	-0.012	-0.091	-0.143

y as a function of x and k^2 for Δ tapered blade

FREQ k^2 / STA	1.1	1.3	1.5	2	3	4	5	6	8	10	12	16	18	21
1	-0.063	-0.158	-0.240	-0.221	-0.144	-0.018	0.074	0.105	0.052	-0.033	-0.068	0.003	0.038	0.041
0.9	0.040	-0.056	-0.140	-0.184	-0.181	-0.088	0.010	0.074	0.090	0.024	-0.040	-0.045	-0.013	0.035
0.8	0.143	0.048	-0.035	-0.122	-0.189	-0.144	-0.061	0.013	0.090	0.071	0.014	-0.056	-0.053	-0.010
0.7	0.247	0.158	0.075	-0.042	-0.164	-0.172	-0.122	-0.059	0.051	0.085	0.063	-0.022	-0.049	-0.050
0.6	0.353	0.269	0.191	0.060	-0.108	-0.166	-0.160	-0.121	-0.019	0.056	0.081	0.038	0.000	-0.040
0.5	0.458	0.384	0.313	0.177	-0.015	-0.118	-0.157	-0.158	-0.096	-0.017	0.041	0.077	0.060	0.020
0.4	0.565	0.501	0.440	0.313	0.112	-0.019	-0.099	-0.141	-0.148	-0.104	-0.043	0.048	0.070	0.073
0.3	0.673	0.622	0.573	0.462	0.277	0.135	0.030	-0.045	-0.128	-0.147	-0.131	-0.053	-0.013	0.035
0.2	0.781	0.745	0.710	0.628	0.479	0.353	0.246	0.157	0.022	-0.065	-0.115	-0.141	-0.131	-0.101
0.1	0.890	0.871	0.853	0.807	0.720	0.639	0.564	0.494	0.370	0.265	0.177	0.041	-0.009	-0.066
0.05	0.940	0.935	0.926	0.902	0.855	0.810	0.767	0.724	0.645	0.570	0.501	0.379	0.324	0.251

Solution (B) "Generalised elliptical" mass taper

Blade mass per unit length $\quad m = (1 - x^2)^n$

Centrifugal force $\quad \frac{g}{w^2} P = \int_x^1 x\,(1 - x^2)^n dx$

$$= \frac{1}{2(n+1)} (1 - x^2)^{n+1}$$

The general differential equation (Ref.)

$$\frac{d}{dx}\left(P \frac{dy}{dx}\right) = -\frac{mp^2}{g} y$$

becomes

$$\frac{1}{2(n+1)} \frac{d}{dx}\left\{(1 - x^2)^{n+1} \frac{dy}{dx}\right\} + k^2 (1 - x^2)^n y = 0$$

or

$$\frac{1 - x^2}{2(n+1)} \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + k^2 y = 0$$

Writing $\underline{x}$ for $(1 - x)$ [i.e. transposing origin to blade tip]

$$\frac{x(2 - x)}{2(n+1)} \frac{d^2 y}{dx^2} + (1 - x) \frac{dy}{dx} + k^2 y = 0$$

The solution in ascending powers of $\underline{x}$ is

$$y = a_0 + a_1 x + \ldots\ldots + a_p x^p + \ldots\ldots$$

Where

$$a_p = \frac{n+1}{p(n+p)} \left(\frac{p - 1n + \frac{p(p-1)}{2}}{n+1} - k^2 \right) a_{p-1}$$

Solutions (A) and (B) $\quad m = (1 - x^2)^n$

Deflection $y = A_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_p x^p + \ldots\ldots$

(B)

Where

$$a_1 = -k^2 a_0$$

$$a_2 = (1-k^2)\,\frac{n+1}{2(n+2)}\,a_1$$

$$a_3 = \frac{(n+1)}{3(n+3)}\left(\frac{2n+3}{n+1} - k^2\right) a_2$$

$$a_4 = \frac{n+1}{4(n+4)}\left(\frac{3n+6}{n+1} - k^2\right) a_3$$

$$a_5 = \frac{n+1}{5(n+5)}\left(\frac{4n+10}{n+1} - k^2\right) a_4$$

$$a_6 = \frac{n+1}{6(n+6)}\left(\frac{5n+15}{n+1} - k^2\right) a_5$$

$$a_7 = \frac{n+1}{7(n+7)}\left(\frac{6n+21}{n+1} - k^2\right) a_6$$

$$a_8 = \frac{n+1}{8(n+8)}\left(\frac{7n+28}{n+1} - k^2\right) a_7$$

$$a_9 = \frac{n+1}{9(n+9)}\left(\frac{8n+36}{n+1} - k^2\right) a_8$$

$$a_{10} = \frac{n+1}{10(n+10)}\left(\frac{9n+45}{n+1} - k^2\right) a_9$$

(A) $(n = \frac{1}{2})$

$$a_1 = -k^2 a_0$$

$$a_2 = \frac{3}{10}(1 - k^2)\,a_1$$

$$a_3 = \frac{1}{7}\left(\frac{8}{3} - k^2\right) a_2$$

$$a_4 = \frac{1}{12}(5 - k^2)\,a_3$$

$$a_5 = \frac{3}{55}(8 - k^2)\,a_4$$

$$a_6 = \frac{1}{26}\left(\frac{35}{3} - k^2\right) a_5$$

$$a_7 = \frac{1}{35}(16 - k^2)\,a_6$$

$$a_8 = \frac{3}{136}(21 - k^2)\,a_7$$

$$a_9 = \frac{1}{57}\left(\frac{80}{3} - k^2\right) a_8$$

$$a_{10} = \frac{1}{70}(33 - k^2)\,a_9$$

Elliptical Taper (Solution A)

Frequency	Mode
$k^2 = 1$	$y = 1 - x$
$k^2 = \frac{8}{3}$	$y = 1 - \frac{8}{3}x + \frac{4}{3}x^2$
$k^2 = 5$	$y = 1 - 5x + 6x^2 - 2x^3$
$k^2 = 8$	$y = 1 - 8x + 16{\cdot}8x^2 - 12{\cdot}8x^3 + 3{\cdot}2x^4$
$k^2 = \frac{35}{3}$	$y = 1 - \frac{35}{3}x + \frac{112}{3}x^2 - 48x^3 + \frac{80}{3}x^4 - \frac{16}{3}x^5$
$k^2 = 16$	$y = 1 - 16x + 72x^2 - \frac{260}{7}x^3 + \frac{880}{7}x^4 - \frac{384}{7}x^5 + \frac{64}{7}x^6$

The above are natural modes (symmetrical or anti-symmetrical) for a complete two-bladed rotor. The anti-symmetrical modes ($k^2 = 1, 5, \frac{35}{3} \ldots$) apply also to a single blade hinged at the axis of rotation ($x = 1$).

For other hinge positions in general we have intermediate values of k^2, leading to a non-terminating series:

e.g. $k^2 = 1{\cdot}5$

$$y = 1 - 1{\cdot}5x + 0{\cdot}225x^2 + 0{\cdot}0375x^3 + 0{\cdot}01090x^4 + 0{\cdot}00388x^5 + 0{\cdot}00152x^6 + 0{\cdot}00063x^7$$

This is found to have a hinge ($y = 0$) at $x = 0{\cdot}770$

i.e. $0{\cdot}23R$ from the axis

Elliptical Taper, case (A)

Vibration amplitude y as a function of frequency (k^2) and blade station x

x \ $k^2 =$	5	6	7	$\frac{35}{3}$	14	16
1	0	+·123	+·185	0	−·118	−·143
0·9	−·098	+·030	+·124	+·095	−·034	−·110
0·8	−·184	−·072	+·029	+·159	+·066	−·017
0·7	−·246	−·167	−·076	+·169	+·145	+·080
0·6	−·272	−·238	−·174	+·113	+·162	+·150
0·5	−·250	−·263	−·242	0	+·097	+·143
0·4	−·168	−·225	−·249	−·137	−·039	+·037
0·3	−·014	−·103	−·166	−·233	−·188	−·131
0·2	+·224	+·126	+·044	−·183	−·224	−·233
0·1	+·558	+·486	+·418	+·161	+·064	−·005
0	+1	+1	+1	+1	+1	+1

Solution (D) "Generalised parabolic" mass taper

Blade mass per unit length $\quad m = (1 - x)^n$

Centrifugal force $\quad \frac{g}{w^2} P = \int^{1} x(1 - x)^n \, dx$

$$= \frac{(1 - x)^{n+1}}{n + 1} - \frac{(1 - x)^{n+2}}{n + 2}$$

$$= \frac{1 + \overline{n + 1}\, x}{(n+1)\,(n+2)} (1 - x)^{n+1}$$

The general differential equation (Ref.)

$$\frac{d}{dx}\left(P \frac{dy}{dx}\right) = - \frac{mp^2}{g} y$$

becomes $$\frac{d}{dx}\left\{\frac{(1-x)(1+\overline{n+1}x)}{(n+1)\,(n+2)} (1 - x)^n \frac{dy}{dx}\right\} + k^2 y\,(1-x)^n = 0$$

or $$\frac{1 + nx - \overline{n+1}\, x^2}{(n+1)(n+2)} \frac{d^2y}{dx^2} - x\frac{dy}{dx} + k^2 y = 0$$

Writing $\underline{x}$ for $(1 - x)$ [i.e. transposing origin to blade tip]

$$x\left(\frac{1}{n + 1} - \frac{x}{n + 2}\right) \frac{d^2y}{dx^2} + (1 - x) \frac{dy}{dx} + k^2 y = 0$$

The solution in ascending powers of $\underline{x}$ is

$$y = a_o + a_1 x + \ldots\ldots + a_p x_p + \ldots\ldots$$

Where $$a_p = \frac{n + 1}{p(n + p)} \left(\frac{\overline{p - 1}.\overline{p + n}}{n + 2} - k^2\right) a_{p-1}$$

Parabolic taper, case (C)

Blade vibration amplitude y as a function of frequency (k^2) and blade station (x)

x \ k^2 =	4	5·4	7	8·8	11	13	15
1	-·119	+·080	+·182	+·143	+·004	-·099	-·132
0·9	-·198	-·006	+·131	+·174	+·096	-·011	-·092
0·8	-·250	-·114	+·047	+·149	+·155	+·085	-·008
0·7	-·269	-·195	-·070	+·077	+·154	+·148	+·095
0·6	-·248	-·246	-·157	-·026	+·095	+·146	+·151
0·5	-·182	-·249	-·229	-·145	-·026	+·066	+·125
0·4	+·066	-·189	-·241	-·227	-·155	-·071	+·008
0·3	+·106	-·050	-·162	-·221	-·233	-·203	-·154
0·2	+·339	+·184	+·045	-·069	-·161	-·208	-·227
0·1	+·635	+·528	+·419	+·308	+·193	+·104	+·029
0	+1	+1	+1	+1	+1	+1	+1

Solution (D) extended

The mass distribution $m = (1 - x)^n$ is unsymmetrical about the origin $(x = 0)$ so that a different blade taper is represented by the expression with x negative. Convenient solutions for the vibration modes can be obtained for these 'blade extensions' also, and these are of practical value, particularly for $n < 0$, since the corresponding taper curve is hollow and has a finite tip ordinate.

The length of the extension has to be chosen to make $P = 0$ at the new blade tip, i.e. the origin $(x = 0)$ has to be at the c.g. of the total mass. From the expression for P already derived, the length of the extension is seen to be $\frac{1}{n+1}$, i.e. $x = -\frac{1}{n+1}$ at the new blade tip.

A further transposition of the origin to this point is found desirable so as to give convergence of the series solution. The new differential equation is

$$x\left(\frac{1}{n+1} - \frac{x}{n+2}\right)\frac{d^2y}{dx^2} + \left(\frac{1}{n+1} - x\right)\frac{dy}{dx} + k^2 y = 0$$

and the solution in ascending powers of x is

$$y = a_0 + a_1 x + \ldots\ldots a_p x^p + \ldots\ldots$$

with $$a_p = \frac{n+1}{p^2}\left(\frac{\overline{p-1}\;\overline{n+p}}{n+2} - k^2\right)a_{p-1}$$

Case (D), extended (=E)

The "modified parabolic" mass distribution $(1 - x)^n$ with n ranging from 0 to 1 covers the uniform $(n = 0)$ and uniformly-tapered blade $(n = 1)$ and as intermediate cases parabolas of various orders, including the common parabola $(n = \frac{1}{2})$.

When n is negative the mass becomes infinite at the blade tip $(x = 1)$ but the solution remains valid and useful for negative values of x, i.e. for the blade continuation beyond the axis. The special value of this result is that the mass distribution curve is now concave, thus more resembling actual blades, and further it has a finite ordinate at the blade extension tip.

The latter is given by
$$(1 - x) = +\left(\frac{n+2}{n+1}\right)$$
$$x = -\left(\frac{1}{n+1}\right)$$

This is the distance from the axis at which the centrifugal force vanishes;; in other words the axis of rotation is at the centroid of the blade plus its extension. The tip ordinate is $\left(\frac{n+2}{n+1}\right)^n$.

The taper and concavity range is seen from the table.

Mass curve	$n = 0$	$-\frac{1}{2}$	$-\frac{3}{4}$	(-1)
Root ordinate	1	1	1	(∞)
Mid ordinate	1	·707	·438	(2α)
Tip ordinate	1	·577	·299	(α)
Area coefficient	1	·732	·495	0
Centroid from axis	0·5	·455	·404	0

The limiting case $(n = -1)$, which requires to be solved separately, is also quoted for comparison.

Hyperbolic mass distribution (E^1)

Blade mass per unit length $= x^{-1}$ (x from axis)

or $(1 - x)^{-1}$ (x from tip)

This can be regarded as a special case of the extended solution (D), with n = -1, but this gives an infinite blade length A solution for finite blade length (= 1) is however readily obtained, namely, in ascending powers of distance x from the tip.

$$y = a_0 + a_1 x + \ldots\ldots + a_p x^p + \ldots\ldots$$

$$\text{where} \quad a_p = \left\{\frac{(p-1)^2 - k^2}{p^2}\right\} a_{p-1}$$

The fact that the mass grading becomes infinite at the axis means that the length of blade to which the solution can be applied has to be cut short of this point.

Regarded as a special case of the extended solution (D) it is seen that the hyperbolic mass distribution gives the limiting degree of hollowness of the mass curve to which the solution applies.

The characteristic bend stiffness and shear stiffness

The various exact solutions worked out for rotating flexible blades apply it is found also to non-rotating blades having the same mass distribution and a certain characteristic bend stiffness distribution related to the mass distribution. Similarly for shear stiffness.

It is considered sufficient to work out one typical example.

For case (D), mass distribution $(1 - x)^n$ ($\underline{x}$ from root) and for the first non-rigid mode, with $k^2 = \frac{2(n+3)}{n+2}$ (applicable to the blade proper <u>plus</u> its extension beyond the axis, with both ends free).

The node, on substitution in the general series is

$$y = 1 - \frac{2(n+3)}{n+2}x + \frac{(n+1)(n+4)(n+3)}{(n+2)^3}x^2 \quad (\underline{x} \text{ from tip})$$

The shear force is proportional to

$$\int yx^n\,dx = \frac{1}{n+1}x^{n+1} - \frac{2(n+3)}{(n+2)^2}x^{n+2} + \frac{(n+1)(n+4)}{(n+2)^3}x^{n+3}$$

By further integration, the bending moment is

$$\frac{1}{(n+1)(n+2)}x^{n+2} - \frac{2}{(n+2)^2}x^{n+3} + \frac{n+1}{(n+2)^3}x^{n+4}$$

The curvature being constant, this is also the required bend stiffness

$$EI = x^{n+2}\left(\frac{\quad}{n+2} - \frac{\quad}{n+1}x\right)^2 \text{ apart from a constant multiplier.}$$

Similarly, the shear stiffness got by dividing the shear force by the slope $\frac{dy}{dx}$ is

$$GA = x^{n+1}\left(\frac{\quad}{n+2} - \frac{\quad}{n+1}x\right) \text{ apart from a constant multiplier.}$$

Analagous results are found for all the types of mass distribution investigated, and these are found to be general and applicable to all modes.

Hinge positions versus k^2

Uniform

	k^2	x
0th	1	0·000
	1·1	0·063
	1·3	0·167
	1·5	0·250
1th	6	0·000
	7	0·112
	8	0·207
2nd	15	0·000
	16	0·046
	18	0·134
	20	0·207
3rd	29	0·000
	33	0·126
	36	0·177

Δ Tapered

	k^2	x
0th	1	0·000
	1·3	0·152
	1·5	0·231
	2	0·342
1th	4	-0·014
	5	0·115
	6	0·222
2nd	10	0·055
	12	0·176
3rd	16	0·002
	18	0·071
	21	0·182

Application of results to fixed-root blades

The results for anti-symmetrical vibration modes ($y = 0$ at $x = 1$) of a rotor consisting of two similar blades (e.g. Cases A, B) and for the corresponding modes in other cases, can be regarded as applying equally to the anti-symmetric vibration when the two blades are rigidly connected to each other *via* a rigid hub, subject to possible restraining effects on the hub from its mounting on the vertical driving shaft (i.e. the hub may not be freely pivoted, as assumed, at the axis of rotation).

However, we are interested also in applying the results, as far as feasible, to the symmetrical vibrations of such rigid-hub type rotors, when the root condition for the vibration amplitude or blade deflection is $\frac{dy}{dx} = 0$. If the hub itself is sufficiently rigid, this root condition may be applied at the station where the blade root enters the hub.

Such a condition, in conjunction with $y = 0$ (which will be true if the blade mass is small enough in relation to that of the rest of the helicopter, and the rotor mounting rigid) cannot be met by any curve applicable to a perfectly flexible blade, as here assumed, but an equivalent hinge position, where the condition $y = 0$ only has to be met, can be determined, making use of the fact that the restraining effect of the hub on the blade bending extends only over a limited distance, which can be estimated. It is in fact $\sqrt{EI/P}$ very nearly, as for a blade of uniform stiffness (EI) and constant tension (P); this is the distance of the equivalent hinge (or effective node) from the point where the blade enters the hub. This is the actual distance, to be expressed as a fraction of the rotor radius R in accordance with the terminology here used.

Rigid mounting of the hub in the vertical direction ($y = 0$) has been assumed but if this condition is not fulfilled the mode position will be changed accordingly, by an amount dependent on the spring constant of the mounting and the mass of the hub in relation to that of the blade. In general the effective node will tend to move radially outward to some extent. This applies for instance when the mounting is flexible enough to leave the hub substantially free

LIST OF DIAGRAMS

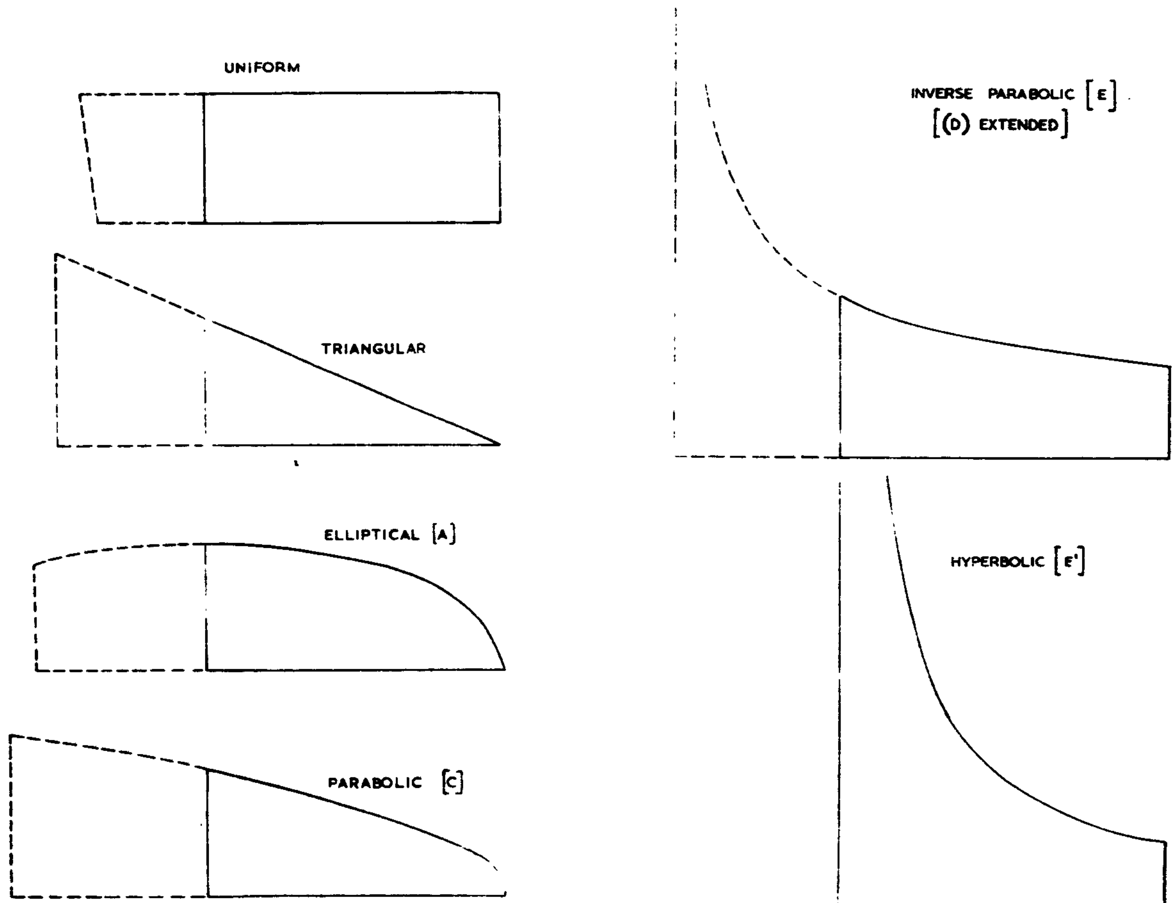

FIG. 1. MASS DISTRIBUTION CURVES

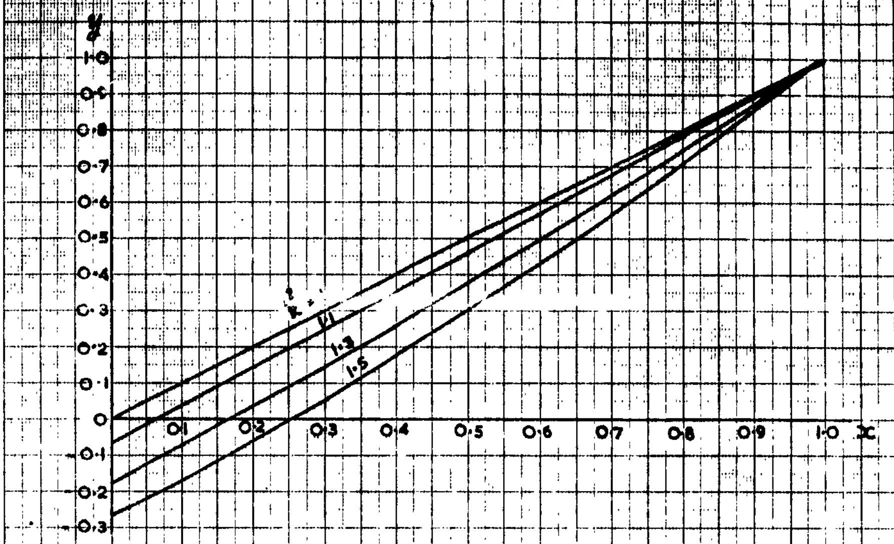

FIG. 2. UNIFORM BLADE O[TH] MODE

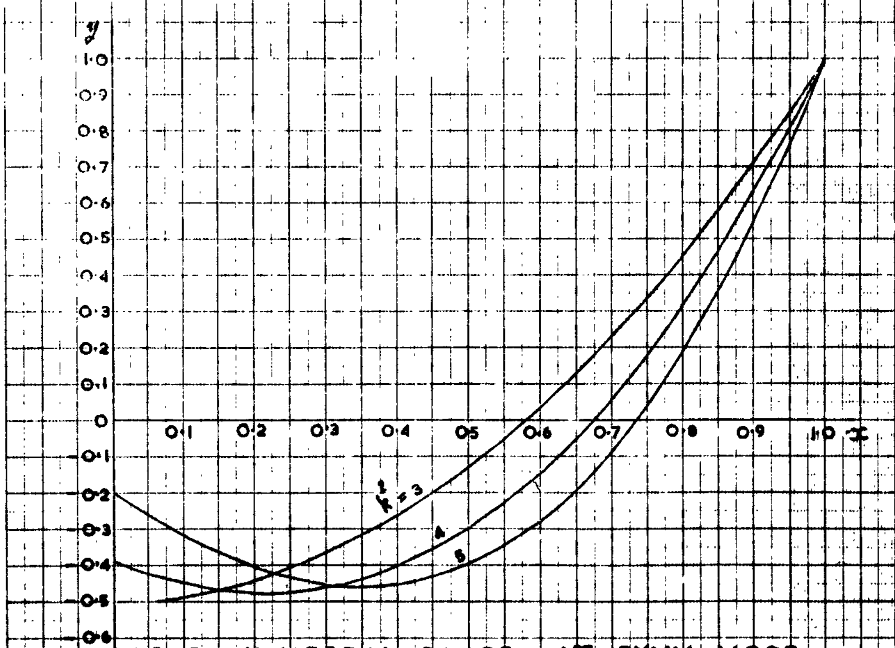

FIG. 3. UNIFORM BLADE 1[ST] SYMM MODE.

Fig. 4

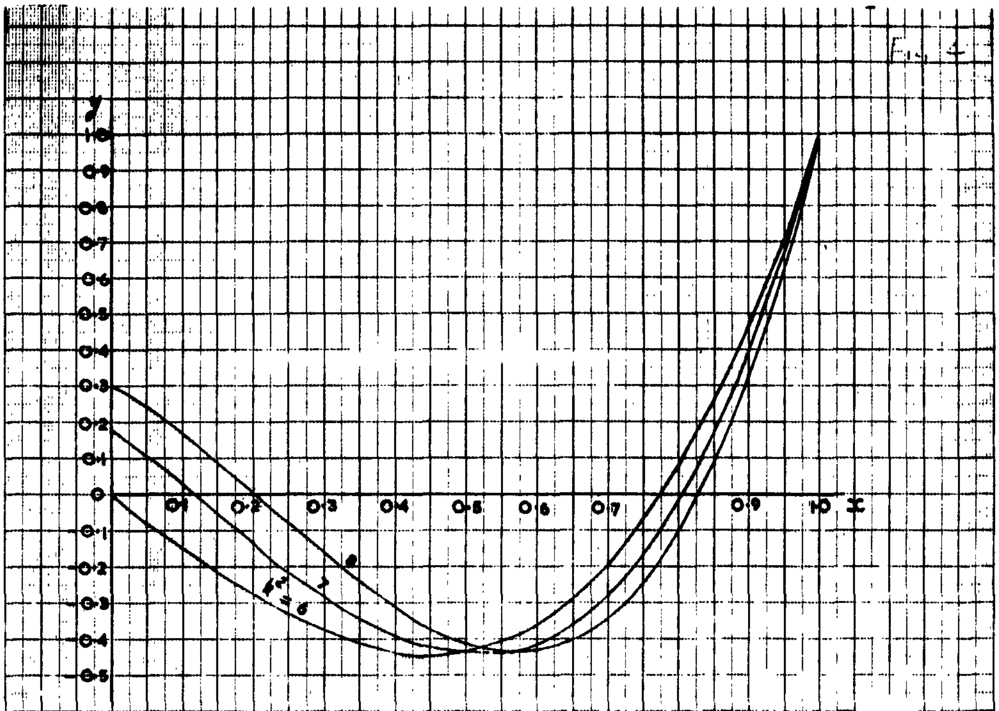

FIG. 4. UNIFORM BLADE 1ST MODE.

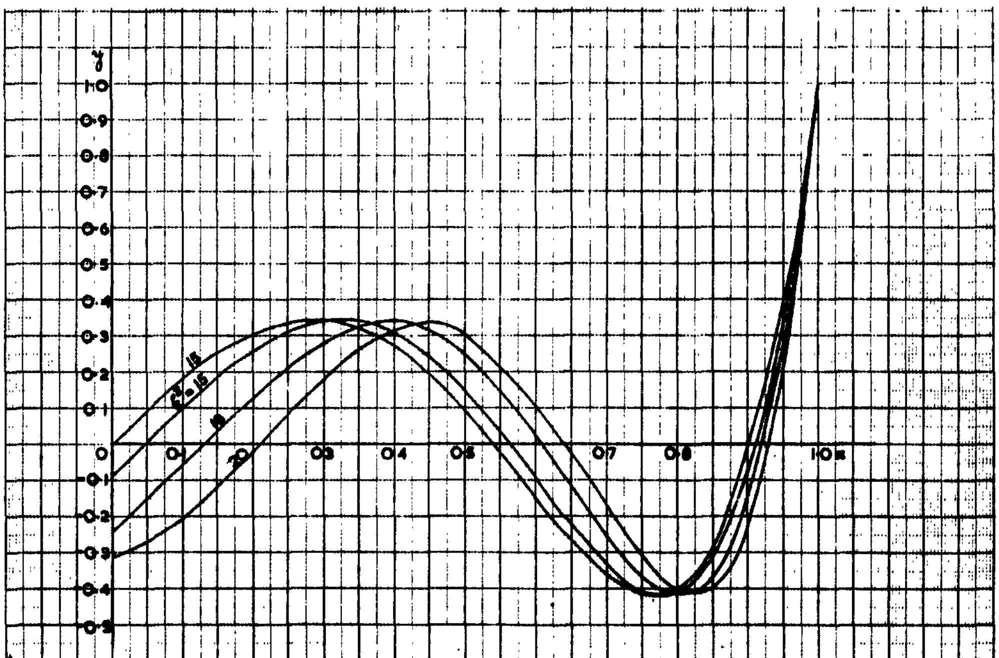

FIG. 5. UNIFORM BLADE 2nd MODE.

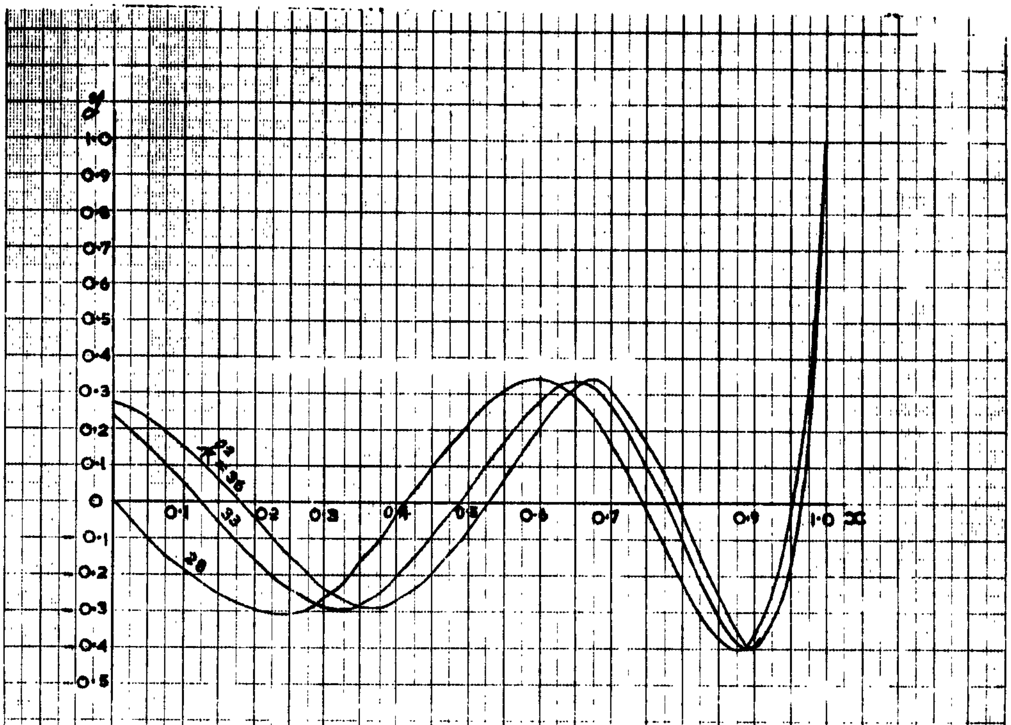

FIG. 6. UNIFORM BLADE 3RD MODE.

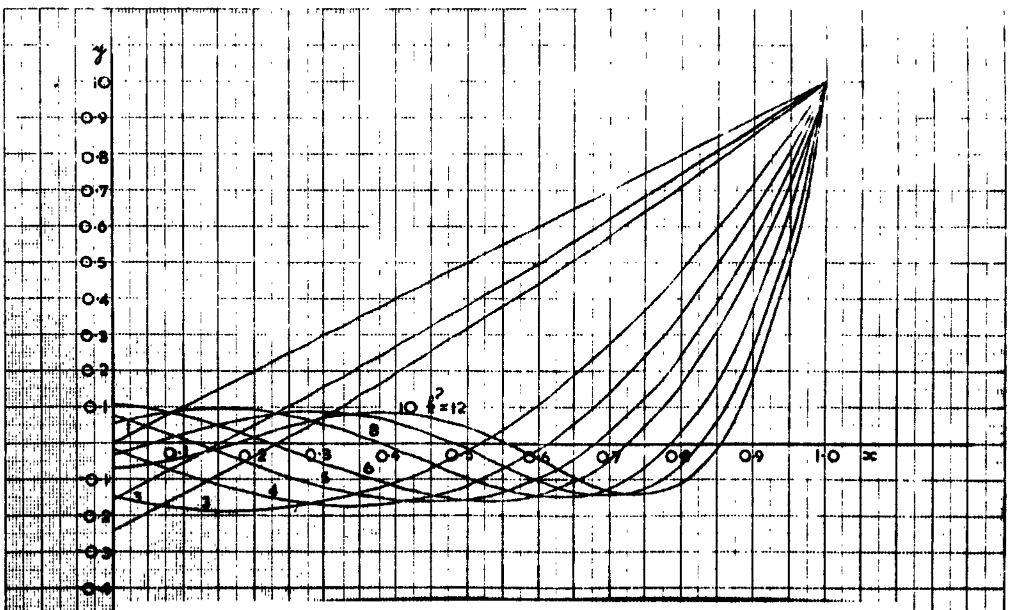

FIG. 7. LINEARLY TAPERED BLADE, VARIOUS MODES.

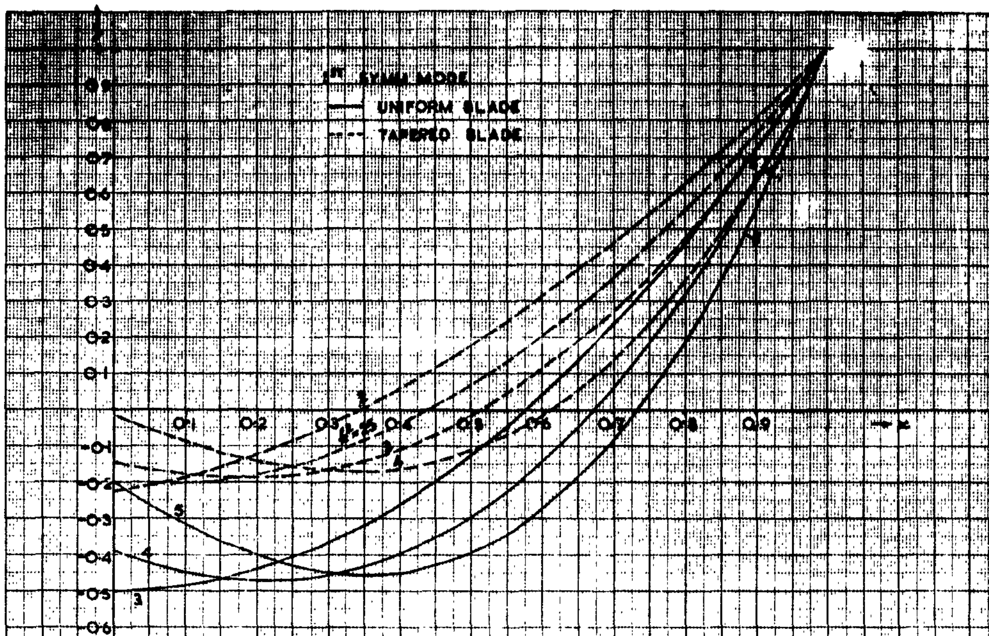

FIG. 8. 1ST SYMM MODE UNIFORM AND TAPERED BLADE.

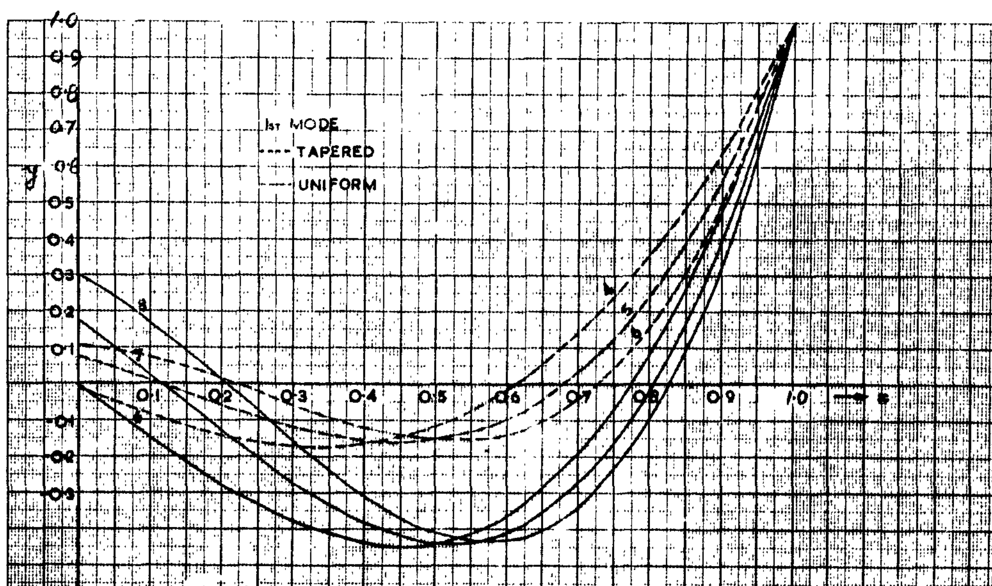

FIG. 9. 1ST MODE UNIFORM AND TAPERED BLADE.

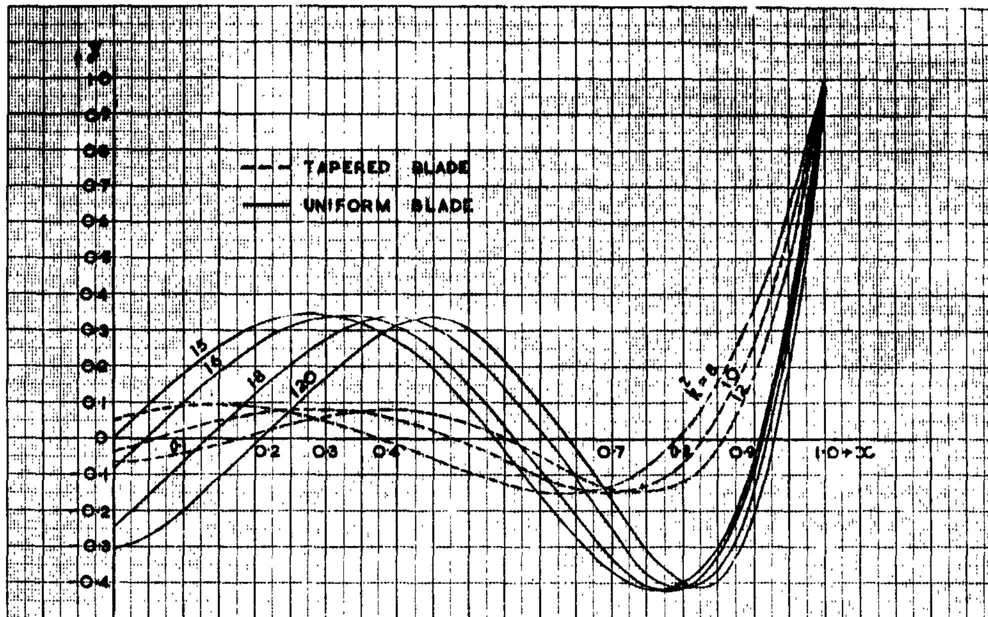

FIG.10. 2ND MODE, UNIFORM AND TAPERED BLADE.

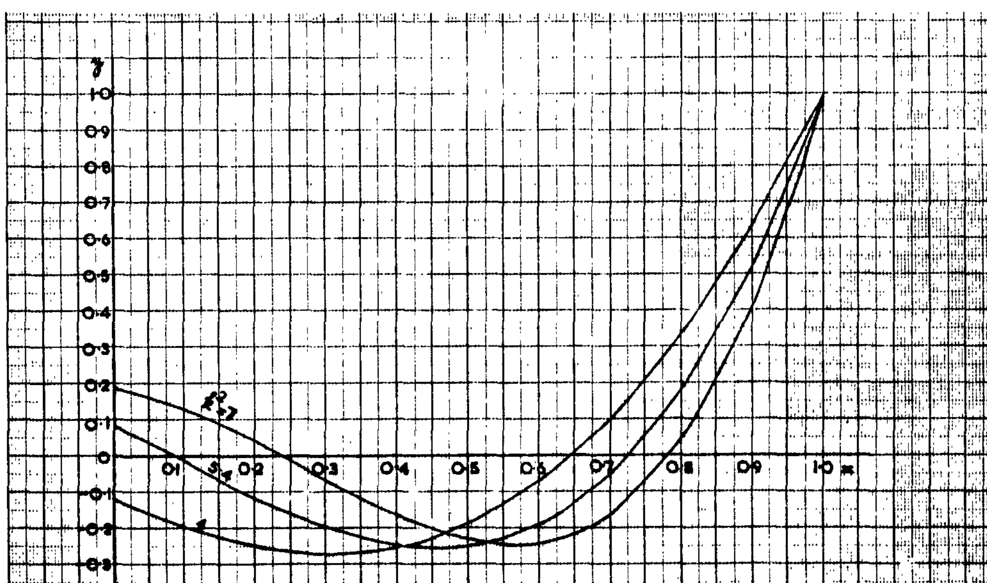

FIG.11. PARABOLIC BLADE. 1ST MODE.

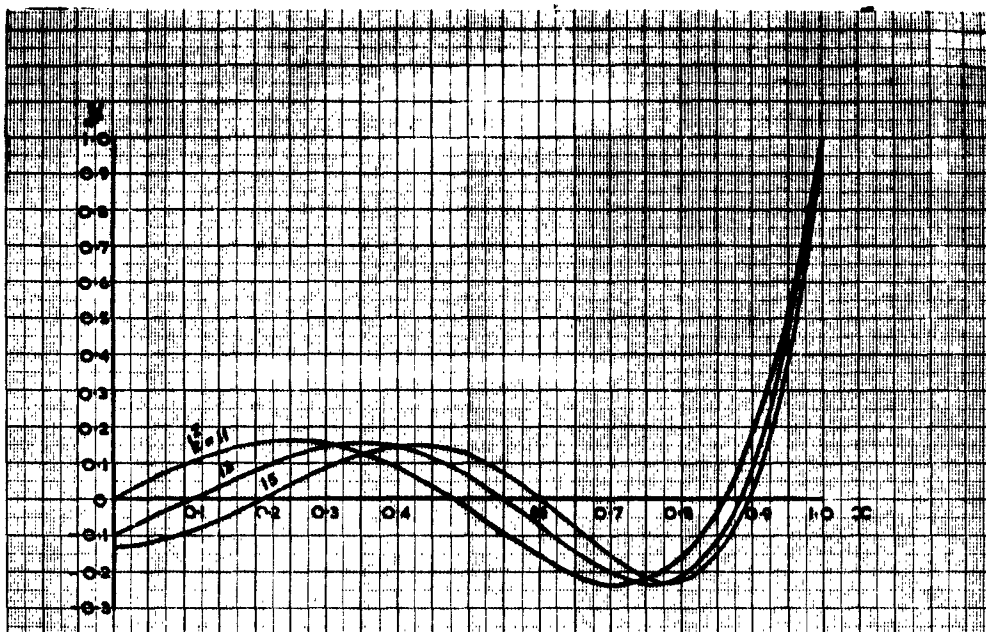

FIG. 12. PARABOLIC BLADE. 2ND MODE

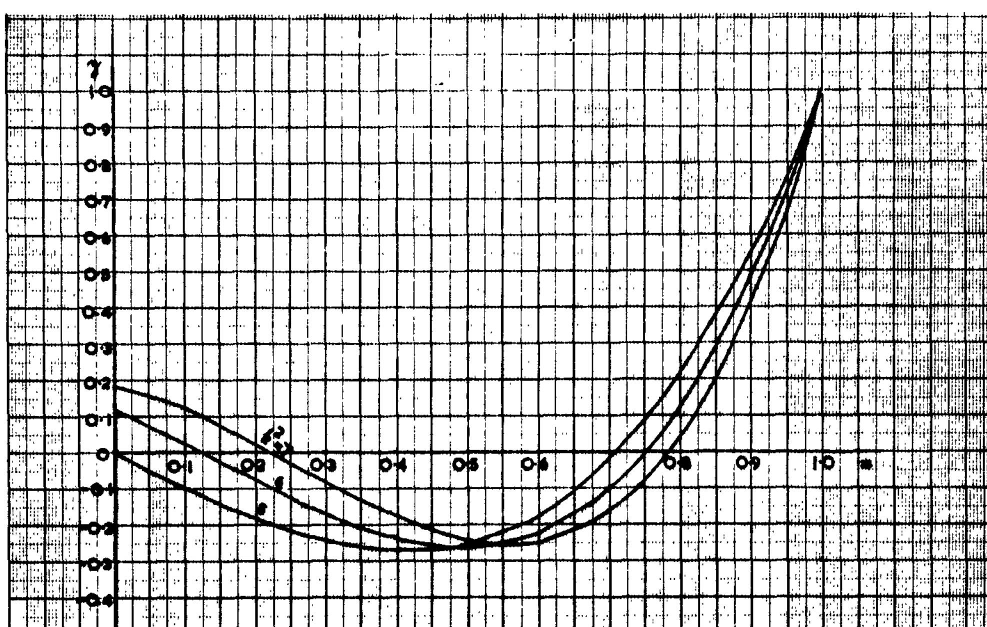

FIG. 13. ELLIPTICAL BLADE. 1ST MODE

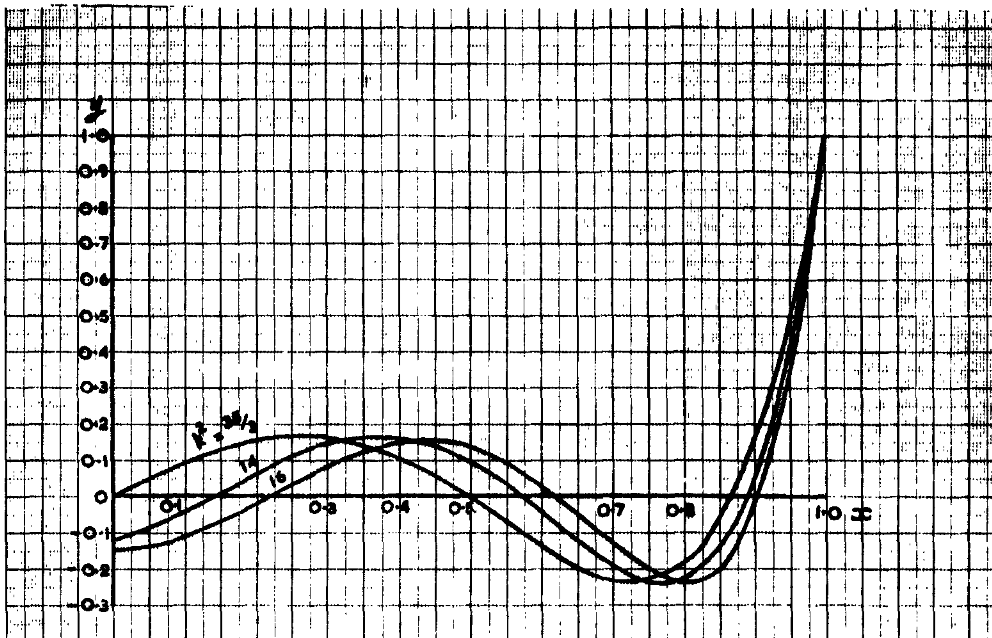

FIG. 14. ELLIPTICAL BLADE 2ND MODE.

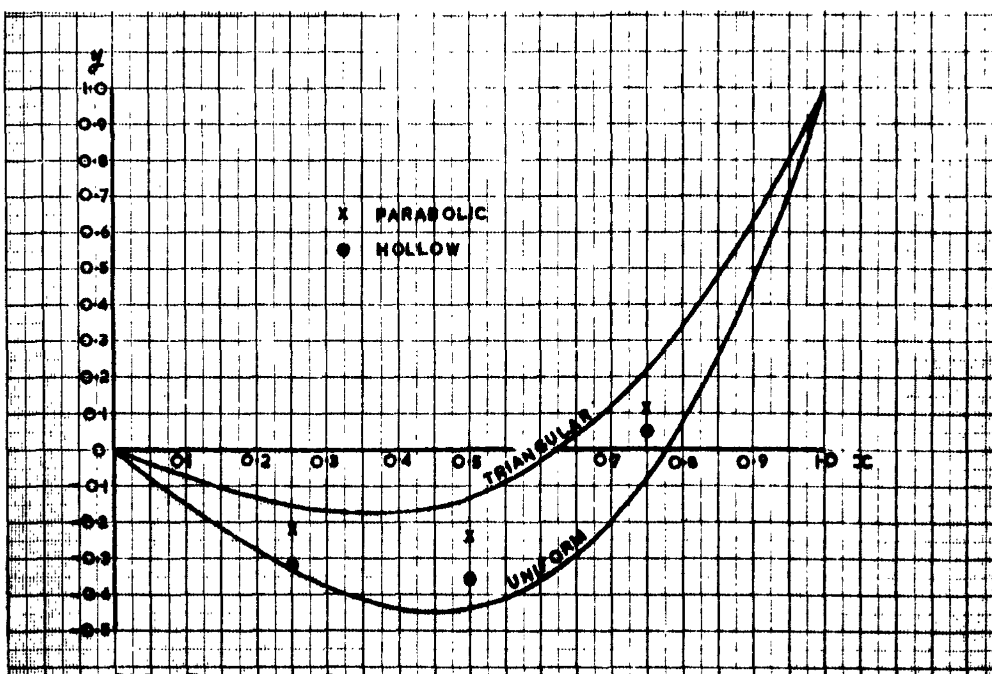

FIG. 15. VARIOUS TAPERS 1ST MODE